Computation of conjugate times in smooth optimal control: the COTCOT algorithm

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***Abstract*— Conjugate point type second order optimality conditions for extremals associated to smooth Hamiltonians are evaluated by means of a new algorithm. Two kinds of standard control problems fit in this setting: the so-called regular ones, and the minimum time singular single-input affine systems. Conjugate point theory is recalled in these two cases, and two applications are presented: the minimum time control of the Kepler and Euler equations.**

I. INTRODUCTION

We consider a smooth Hamiltonian equation

$$\dot{z} = \overrightarrow{H}(z) \tag{1}$$

on the cotangent bundle of a smooth manifold M. Such an equation arises in the optimal control of systems with smooth control. Indeed, extremal trajectories are parameterized by Pontryagin maximum principle and satisify the standard Hamiltonian equation. In the two cases of *regular* systems, and *singular* single-input affine minimum time systems, the control is smooth and a Hamiltonian equation of the form (1) is derived. Moreover, second order conditions for (local) optimality of a given extremal, z, can be checked by computing a set of solutions to the variational system along the extremal:

$$\delta\dot{z} = d\overrightarrow{H}(z(t))\delta z. \tag{2}$$

System (2) is called the *Jacobi* equation. This kind of second order conditions are known as *conjugate point* conditions [1], [2], [3]. An implementation of the relevant computations, including solving (1) and (2) is provided by the *Matlab* package `cotcot` [4]. More precisely, on the basis of a user-provided Hamiltonian, the second members of (1) and (2) are evaluated by automatic differentiation [5]. The numerical integration of the differential equations and the solution of the associated shooting problem are computed by standard *Netlib* codes interfaced with *Matlab*. We propose two applications of the algorithm in spaceflight dynamics: first to orbit transfer, then to attitude control.

To this end, we first recall in §II and §III the conjugate point theory, respectively for regular control problems and minimum time singular single-input affine systems. Then, the minimum time control of the Kepler equation is presented in §IV. The aim is to compute orbit transfers around the Earth and to check optimality of the corresponding extremals. This is done in the regular multi-input case as well as in the singular single-input exceptional case. The second application is the attitude control of a spacecraft. A preliminary study of the Euler equations is achieved. The hyperbolic and exceptional singular cases of the single-input system are finally analyzed in §V. For a more detailed presentation of the topic, we refer readers to [6], [7].

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II. REGULAR CONTROL SYSTEMS

Consider the control of the system

$$\dot{x} = f(x,u),\ x(0) = x_0 \tag{3}$$

where x belongs to a smooth manifold M identified with $\mathbf{R}^n$, and where the cost to minimize is the functional

$$C(x,u) = \int_0^T f^0(x,u)dt.$$

The right hand side $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is smooth and u takes values in $\mathbf{R}^m$. Since the control domain is unbounded, every optimal control u on $[0,T]$ is a singularity of the endpoint mapping $E_{x_0,t} : \mathrm{L}^\infty_m([0,t]) \to \mathbf{R}^n$ for $0 < t \le T$ where $E_{x_0,t}(u) = x(t,x_0,u)$ is the solution of (3): the Fréchet derivative at u of the mapping is not surjective (its image has codimension at least one; see assumption **(A2)** hereafter). The resulting trajectory is the projection of an extremal (x,p^0,p,u), p^0 non-positive, solution of the maximum principle,

$$\dot{x} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial x}$$

and

$$\frac{\partial H}{\partial u} = 0$$

where $H = p^0 f^0(x,u) + \langle p, f(x,u)\rangle$ is the standard Hamiltonian, constant along the extremal, zero if the final time is free. The Hamiltonian is homogeneous in (p^0,p) and we have two cases: the *normal* case where p^0 is not zero and normalized to $p^0 = -1$, and the *exceptional* case otherwise, $p^0 = 0$. Without losing any generality, we may assume that the trajectory is one to one on $[0,T]$. We make the strong Legendre assumption,

(A1) The quadratic form $\partial^2 H/\partial u^2$ is negative definite along the reference extremal.

Therefore, using the implicit function theorem, the extremal control can be locally defined as a smooth function u_r of