# Optimal distribution coefficients for packets traffic on a telecommunication network

Alessia Marigo
Dept. of Math., University of Roma ”La Sapienza”, Roma, Italy. marigo@mat.uniroma1.it
Benedetto Piccoli
I.A.C.-C.N.R., Roma, Italy. piccoli@iac.rm.cnr.it

***Abstract*— The aim of this paper is to optimize traffic distribution coefficients in order to maximize the trasmission speed of packets over a network. We consider a macroscopic fluidodynamic model dealing with packets flow proposed in [10], where the dynamics at nodes (routers) is decided by a routing algorithm depending on traffic distribution (and priority) coefficients. We solve completely the problem for the simple case of two incoming and two outgoing lines.**

## I. INTRODUCTION.

There are some recent works on car traffic flow on networks, see [7], [8], [11], that relie on macroscopic description via car densities and other conserved quantities[3], [12], [13]. To treat a telecommunication network as Internet, we look at at an intermediate time scale, thus assume that packets transmission happens at a faster level but the equilibria of the whole network are reached only as asymptotic.
A network is formed by a finite collection of transmission lines and nodes (or routers), each packet is seen as a particle on the network and it is assumed that:

1) Each packet travels on the network with a fixed speed and with assigned final destination;
2) Routers receive, process and then forward packets. Packets may be lost with a probability increasing with the number of packets to be processed. Each lost packet is sent again.

Based on these rules, first the behavior of a single straight transmission line is modeled. Each router sends packets to the following one a first time, then packets that are lost in this process are sent a second time and so on. The important point is that packets are sent until they reach next router, thus, looking at intermediate time scale, it is assumed that packets are conserved and we consider a simple model consisting of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \tag{1}$$

where $\rho$ is the packet density, $v$ is the velocity and $f(\rho) = v\rho$ is the flux. Since the speed on the line is assumed constant, we can derive an average transmission speed among routers considering the amount of packets that may be lost. Assigning a loss probability as function of the density, it is possible to pass to the limit in the (re)sending procedure getting a velocity function and thus a flux function. Since the aim is to consider complex networks, we need to introduce a way of solving dynamics at nodes in which many lines (backbones) intersect. For this, we propose the following routing algorithm:

(RA) Packets are processed by arrival time and are sent to outgoing lines in order to maximize the flux.

A key role is played by Cauchy problems with initial data constant on each transmission lines called Riemann problems at the node. In order to determine unique solutions to Riemann problems, some additional parameters are introduced, called respectively priority parameters and traffic distribution parameters. The theory for this model is developed in [10].

Then we focus on a simple network formed of a single junction with two incoming and two outgoing lines. We assume that packets flow from two initial nodes to two final ones. Assigned the packet quantities flowing from initial to final nodes, we compute the final equilibrium as function of the traffic distribution (and priority) parameter. Such equilibrium determines the average speeds at which packets travel on the network and we define some functional measuring the average travel time. The aim is to optimize the choice of the traffic distribution parameter in order to minimize such functionals. The problem is completely solved giving the optimal values as function of the packets densities. It is interesting to notice that in many cases there is a set of opitmal values (with the extreme case of functional not depending on the parameter).

The paper is organized as follows. Section 2 describes the dynamics of packet density on a single transmission line giving two examples. Section 3 gives basic definitions and notation for telecommunication networks. Section 4 illustrates the routing algorithm for Riemann problems at junctions. Finally, in Section 5 we compute the optimal parameters for the examples of section 2 and a simple network.

## II. PACKETS LOSS AND VELOCITY FUNCTIONS ON TRANSMISSION LINES.

Each transmission line, represented by a real interval $I$, consists of many edges and nodes. Each node corresponds to a server sending and receiving packets. To determine the dynamics on $I$ we need to describe the effect of packets loss on the velocity of transmission function. As for the

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Internet, we assume that each node $N_k$ sends again packets that are lost by the following node $N_{k+1}$. More precisely, we assume that there exists a function $p : [0, \rho_{max}] \mapsto [0, 1]$ that assigns the packet loss probability as function of the packet density. Suppose that $\delta$ is the distance between the nodes $N_k$ and $N_{k+1}$. Let $\Delta t_0$ be the transmission time of packets from node $N_k$ to node $N_{k+1}$ in the case in which they are sent with success at the first attempt, and $\Delta t_{av}$ the average transmission time when some packets are lost by $N_{k+1}$ and they are sent again by $N_k$. Let us denote with $\bar{v} = \frac{\delta}{\Delta t_0}$ and $v = \frac{\delta}{\Delta t_{av}}$ the packets velocity, respectively, in the two cases. Therefore at the first attempt the packets sent by node $N_k$ reach with success node $N_{k+1}$ with probability $(1-p)$ and they are lost by node $N_{k+1}$ and sent again by node $N_k$ with probability $p$. At the second attempt there are $(1 - p)$ packets to be sent again and $(1 - p)p$ packets are sent with success while $p^2$ are lost. Going on at the $n$-th attempt $(1 - p)p^{n-1}$ packets are sent successfully and $p^n$ are lost. The average transmission time is equal to $\Delta t_{av} = \sum_{n=1}^{+\infty} n \Delta t_0 (1 - p)p^{n-1} = \frac{\Delta t_0}{1-p}$, from which we get that the transmission velocity is given by $v = \frac{\delta}{\Delta t_{av}} = \frac{\delta}{\Delta t_0}(1 - p) = \bar{v}(1 - p)$. Once packets loss has been measured, then the corresponding flux is easily determined.

*Example 1:* Let us suppose that the packets loss probability is given by

$$p(\rho) = \begin{cases} 0, & 0 \le \rho \le \sigma, \\ \frac{2(\rho-\sigma)}{\rho}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

The transmission velocity is equal to

$$v(\rho) = \bar{v}(1 - p(\rho)) = \begin{cases} \bar{v}, & 0 \le \rho \le \sigma, \\ \bar{v}\frac{(2\sigma-\rho)}{\rho}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

Imposing that $v(\rho_{\max}) = \bar{v}\frac{(2\sigma-\rho_{\max})}{\rho_{\max}} = 0$, we get that $\sigma = \frac{\rho_{\max}}{2}$. Since $f(\rho) = v(\rho)\rho$ it follows that

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \bar{v}(2\sigma - \rho), & \sigma \le \rho \le \rho_{\max}. \end{cases} \tag{2}$$

*Example 2:* If the packets loss probability is given by $p(\rho) = \frac{\rho+\bar{v}-1}{\bar{v}}$, then the transmission velocity is $v(\rho) = 1-\rho$ and the flux function is

$$f(\rho) = \rho(1 - \rho). \tag{3}$$

In what follows we suppose that measures on packets loss probability leads to the formulation of Example 1 or 2. Observe that for Example 1 the corresponding flux has the property that $f'(\sigma^{\pm}) \neq 0$ that allows to control the variation of the density function in terms of the variation of the flux function. We suppose for simplicity that $\rho_{\max} = 1$.

## III. Telecommunication networks.

We consider a telecommunication network, that is modelled by a finite set of intervals $I_i = [a_i, b_i] \subset \mathbb{R}, i = 1, ..., N, a_i < b_i$, on which we consider the equation (1). Hence the datum is given by a finite set of functions $\rho_i$ defined on $[0, +\infty[ \times I_i$.

On each transmission line $I_i$ we want $\rho_i$ to be a weak entropic solution, that is for every function $\varphi : [0, +\infty[ \times I_i \mapsto \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[ \times ]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left( \rho_i \frac{\partial \varphi}{\partial t} + f(\rho_i) \frac{\partial \varphi}{\partial x} \right) dx dt = 0, \tag{4}$$

and for every $k \in \mathbb{R}$ and every $\tilde{\varphi} : [0, +\infty[ \times I_i \mapsto \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[ \times ]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \Big( |\rho_i - k| \frac{\partial \tilde{\varphi}}{\partial t} + sgn(\rho_i - k) \quad (f(\rho_i) - f(k)) \frac{\partial \tilde{\varphi}}{\partial x} \Big) dx dt \ge 0.$$

It is well known that, for equation (1) on $\mathbb{R}$ and for every initial data in $L^\infty$, there exists a unique weak entropic solution depending in a continuous fashion from the initial data in $L^1_{loc}$. Moreover, for initial data in $L^\infty \cap L^1$ we have Lipschitz continuous dependence in $L^1$, see [5], [6].

We assume that the transmission lines are connected by some junctions. Each junction $J$ is given by a finite number of incoming transmission lines and a finite number of outgoing transmission lines, thus we identify $J$ with $((i_1, ..., i_n), (j_1, ... j_m))$ where the first $n$-tuple indicates the set of incoming transmission lines and the second $m$-tuple indicates the set of outgoing transmission lines. Each transmission line can be incoming transmission line at most for one junction and outgoing at most for one junction. Hence the complete model is given by a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I} = \{I_i : i = 1, ..., N\}$ is the collection of transmission lines and $\mathcal{J}$ is the collection of junctions. For boundaries of transmission lines not connected to junctions we can use the theory of [1], [2], [4].

## IV. Riemann problems at junctions.

Now we discuss the solution at junctions. If $\rho = (\rho_1, ..., \rho_{n+m})$ is a weak solution at the junction such that each $x \mapsto \rho_i(t, x)$ has bounded variation, then $\rho$ satisfies the Rankine-Hugoniot condition at the junction $J$, namely

$$\sum_{i=1}^{n} f(\rho_i(t, b_i^-)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, a_j^+)), \tag{5}$$

for almost every $t > 0$. For a scalar conservation law a Riemann problem is a Cauchy problem for an initial data of Heavyside type, that is piecewise constant with only one discontinuity. One looks for centered solutions, i.e. $\rho(t, x) = \phi(\frac{x}{t})$, which are the building blocks to construct solutions to the Cauchy problem via wave front tracking algorithm. These solutions are formed by continuous waves called rarefactions and by travelling discontinuities called shocks. The speed of waves are related to the values of $f'$, see [5], [9]. Analogously, we call Riemann problem for a junction the Cauchy problem corresponding to an initial data which is constant on each transmission line.

To solve Riemann problems according to (RA) we need some additional parameters called priority and traffic distribution parameters. For simplicity of exposition, consider a junction $J$ in which there are two transmission lines with incoming traffic and two transmission lines with outgoing traffic. In this case we have only one priority parameter $p \in\, ]0,1[$ and one traffic distribution parameter $\alpha \in\, ]0,1[$. We denote with $\rho_i(t,x), i=1,2$ and $\rho_j(t,x), j=3,4$ the traffic densities, respectively, on the incoming transmission lines and on the outgoing ones and by $(\rho_{1,0},\rho_{2,0},\rho_{3,0},\rho_{4,0})$ the initial datum. Since the speed of waves must be negative on incoming lines and positive on outgoing ones, we wanto to determine a unique 4-tuple $(\hat{\rho}_1,...,\hat{\rho}_4)\in[0,1]^4$ such that

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup \,]\tau(\rho_{i,0}),1], & \text{if } 0 \le \rho_{i,0} \le \sigma, \\ [\sigma,1], & \text{if } \sigma \le \rho_{i,0} \le 1, \end{cases} \tag{6}$$

$i=1,2$, and

$$\hat{\rho}_j \in \begin{cases} [0,\sigma], & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ \{\rho_{j,0}\} \cup [0,\tau(\rho_{j,0})[\,, & \text{if } \sigma \le \rho_{j,0} \le 1, \end{cases} \tag{7}$$

$j=3,4$, and on each incoming line $I_i$, $i=1,2$, the solution consists of the single wave $(\rho_{i,0},\hat{\rho}_i)$, while and on each outgoing line $I_j$, $j=3,4$, the solution consists of the single wave $(\hat{\rho}_j,\rho_{j,0})$. Define $\gamma_i^{\max}$ and $\gamma_j^{\max}$ as follows:

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0}), & \text{if } \rho_{i,0} \in [0,\sigma], \\ f(\sigma), & \text{if } \rho_{i,0} \in \,]\sigma,1]\,, \end{cases} \quad i=1,2, \tag{8}$$

and

$$\gamma_j^{\max} = \begin{cases} f(\sigma), & \text{if } \rho_{3,0} \in [0,\sigma], \\ f(\rho_{3,0}), & \text{if } \rho_{3,0} \in \,]\sigma,1]\,, \end{cases} \quad j=3,4. \tag{9}$$

The quantities $\gamma_i^{\max}$ and $\gamma_j^{\max}$ represent the maximum flux that can be obtained by a single wave solution on each transmission line. In order to maximize the number of packets through the junction over incoming and outgoing lines we define $\Gamma = \min\{\Gamma_{in},\Gamma_{out}\}$, where $\Gamma_{in} = \gamma_1^{\max}+\gamma_2^{\max}$ and $\Gamma_{out} = \gamma_3^{\max}+\gamma_4^{\max}$. One easily see that to solve the Riemann problem, it is enough to determine the fluxes $\hat{\gamma}_i = f(\hat{\rho}_i), i=1,2$. Let us determine $\hat{\gamma}_i, i=1,2$. We have to distinguish two cases:

**I** $\Gamma_{in}=\Gamma$,

**II** $\Gamma_{in}>\Gamma$.

In the first case we set $\hat{\gamma}_i = \gamma_i^{\max}, i=1,2$. Let us analyze the second case in which we use the priority parameter $p$. Not all packets can enter the junction, so let $C$ be the amount of packets that can go through. Then $pC$ packets come from first incoming line and $(1-p)C$ packets from the second. Consider the space $(\gamma_1,\gamma_2)$ and define the following lines:

$$r_p : \gamma_2 = \frac{1-p}{p}\gamma_1, \quad r_\Gamma : \gamma_1+\gamma_2 = \Gamma.$$

Define $P$ to be the point of intersection of the lines $r_p$ and $r_\Gamma$. Recall that the final fluxes should belong to the region:

$$\Omega = \{(\gamma_1,\gamma_2) : 0 \le \gamma_i \le \gamma_i^{\max}, i=1,2\}.$$

We distinguish two cases:

**a)** $P$ belongs to $\Omega$,

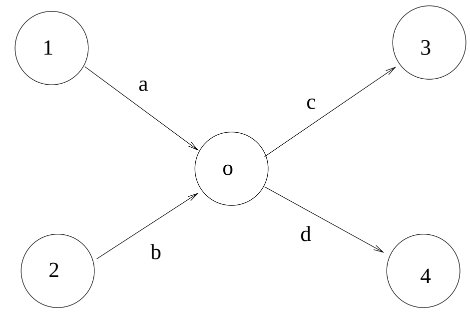


Fig. 1. A simple network.

**b)** $P$ is outside $\Omega$.

In the first case we set $(\hat{\gamma}_1,\hat{\gamma}_2)=P$ , while in the second case we set $(\hat{\gamma}_1,\hat{\gamma}_2)=Q$, with $Q = proj_{r_p\cap r_\Gamma}(P)$ where $proj$ is the usual projection on a convex set.

Let us now determine $\hat{\gamma}_j, j=3,4$. As for the incoming transmission lines we have to distinguish two cases :

**I** $\Gamma_{out}=\Gamma$,

**II** $\Gamma_{out}>\Gamma$.

In the first case $\hat{\gamma}_j = \gamma_j^{\max}, j=3,4$. Let us determine $\hat{\gamma}_j$ in the second case. Recall $\alpha$ the traffic distribution parameter. Since not all packets can go on the outgoing transmission lines, we let $C$ be the amount that goes through. Then $\alpha C$ packets go on the outgoing line $I_3$ and $(1-\alpha)C$ on the outgoing line $I_4$. Consider the space $(\gamma_3,\gamma_4)$ and define the following lines:

$$r_\alpha : \gamma_4 = \frac{1-\alpha}{\alpha}\gamma_3, \quad r_\Gamma : \gamma_3+\gamma_4 = \Gamma.$$

Define $P$ to be the point of intersection of the lines $r_\alpha$ and $r_\Gamma$. Recall that the final fluxes should belong to the region:

$$\Omega = \left\{(\gamma_3,\gamma_4) : 0 \le \gamma_j \le \gamma_j^{\max}, j=3,4\right\}.$$

We distinguish two cases:

**a)** $P$ belongs to $\Omega$

**b)** $P$ is outside $\Omega$.

In the first case we set $(\hat{\gamma}_3,\hat{\gamma}_4)=P$, while in the second case we set $(\hat{\gamma}_3,\hat{\gamma}_4)=Q$, where $Q = proj_{r_\alpha\cap r_\Gamma}(P)$.

We can extend the reasoning to the case of $n$ incoming and $m$ outgoing lines.

## V. Optimization of a simple network

We focus on a simple network as in the Figure 1. There are five nodes $\{1,2,3,4,o\}$ and four edges $\{a,b,c,d\}$, where $a$ and $b$ are the incoming lines to the centre $o$ and $c$ and $d$ are the outgoing lines from the centre $o$. There are packets from nodes $\{1,2\}$ to nodes $\{3,4\}$ passing through $o$ running on lines $a,b,c,d$. We denote them by $c_{ij}$ with $i\in\{1,2\}$ and $j\in\{3,4\}$. Thus the expected packets densities running on the lines are given by

$\rho_a$ from $a$ to $o$: $\rho_a = c_{13}+c_{14}$;

$\rho_b$ from $b$ to $o$: $\rho_b = c_{23}+c_{24}$;

$\rho_c$ from $o$ to $c$: $\rho_c = c_{13}+c_{23}$;

$\rho_d$ from $o$ to $d$: $\rho_d = c_{14}+c_{24}$.

Our aim is first to solve the RP at the junction $o$ assuming the densities on lines are constant and given by the above formula. This gives us the expected equilibrium reached by the network at regime. Then we want to compute the average transmission time at regime over the network as function of the parameters $\alpha$ and $p$. Therefore we introduce the following costs:

$$\begin{array}{l} J_1 = V_{ac} + V_{ad} + V_{bc} + V_{bd} \\ J_2 = c_{ac}V_{ac} + c_{ad}V_{ad} + c_{bc}V_{bc} + c_{bd}V_{bd}, \end{array}$$

where $V_{\varphi\psi} = v(\hat{\rho}_\varphi) + v(\hat{\rho}_\psi)$, $v(\hat{\rho}_\varphi)$, $v(\hat{\rho}_\psi)$ are the velocities on the lines $e_\varphi$, $e_\psi$ and $\hat{\rho}$ is the solution to the RP. We define $\gamma_\varphi^{max}$ (resp. $\gamma_\psi^{max}$) as in equation (8) (resp. equation (9)) and consider the following systems:

$$\begin{cases} \gamma_b = \Gamma - \gamma_a \\ \gamma_b = \frac{1-p}{p}\gamma_a, \end{cases}$$

where $p$ is the priority parameter, and

$$\begin{cases} \gamma_d = \Gamma - \gamma_c \\ \gamma_d = \frac{1-\alpha}{\alpha}\gamma_c, \end{cases}$$

where $\alpha$ is traffic distribution parameter. The points satisfying the above systems are $\tilde{\gamma}_a = p\Gamma, \tilde{\gamma}_b = (1-p)\Gamma$ and $\tilde{\gamma}_c = \alpha\Gamma, \tilde{\gamma}_d = (1-\alpha)\Gamma$, respectively.
Consider the following conditions:
**A1** $\tilde{\gamma}_c = \alpha\Gamma_{in} \le \gamma_c^{max}$;
**A2** $\tilde{\gamma}_d = (1-\alpha)\Gamma_{in} \le \gamma_d^{max}$.
Now, if $\Gamma = \Gamma_{in}$ the solutions given to the RP are the following:
- $(\gamma_a^{max}, \gamma_b^{max}, \tilde{\gamma}_c, \tilde{\gamma}_d)$ if both A1 and A2 are satisfied,
- $(\gamma_a^{max}, \gamma_b^{max}, \gamma_c^{max}, \Gamma_{in} - \gamma_c^{max})$ if A1 is not satisfied and A2 is satisfied,
- $(\gamma_a^{max}, \gamma_b^{max}, \Gamma_{in} - \gamma_d^{max}, \gamma_d^{max})$ if A1 is satisfied and A2 is not satisfied.

Notice that the case of both **A1, A2** false is not possible since otherwise it would be $\Gamma_{in} \ge \Gamma_{out}$.
Consider the following conditions:
**B1** $\tilde{\gamma}_a = p\Gamma_{out} \le \gamma_a^{max}$;
**B2** $\tilde{\gamma}_b = (1-p)\Gamma_{out} \le \gamma_b^{max}$.
Now, if $\Gamma = \Gamma_{out}$ the solutions to the RP are the following:
- $(\tilde{\gamma}_a, \tilde{\gamma}_b, \gamma_c^{max}, \gamma_d^{max})$ if both B1 and B2 are satisfied,
- $(\gamma_a^{max}, \Gamma_{out} - \gamma_a^{max}, \gamma_c^{max}, \gamma_d^{max})$ if B1 is not satisfied and B2 is satisfied,
- $(\Gamma_{out} - \gamma_b^{max}, \gamma_b^{max}, \gamma_c^{max}, \gamma_d^{max})$ if B1 is satisfied and B2 is not satisfied.

Notice that the case of both **B1, B2** is not possible since otherwise it would be $\Gamma_{out} \ge \Gamma_{in}$.
Once fixed $\rho_\varphi$ and $\rho_\psi$, $\varphi \in \{a, b\}$ and $\psi \in \{c, d\}$, we can find for which $\alpha$ and $p$ conditions **A1**, **A2**, **B1**, **B2** are satisfied as follows. If $\Gamma = \Gamma_{in}$, let

$$\gamma_c' = \Gamma - \gamma_d^{max}, \quad \gamma_d' = \Gamma - \gamma_c^{max},$$

$$\beta^- = \frac{\gamma_d'}{\gamma_c^{max}}, \quad \beta^+ = \frac{\gamma_d^{max}}{\gamma_c'},$$

then, for $\alpha \ge \frac{1}{1+\beta^-}$, **A1** is false and **A2** is true, for $\alpha \le \frac{1}{1+\beta^+}$, **A1** is true and **A2** is false and finally, for $\frac{1}{1+\beta^+} \le \alpha \le \frac{1}{1+\beta^-}$, both **A1** and **A2** are true.
If otherwise $\Gamma = \Gamma_{out}$, let

$$\gamma_a' = \Gamma - \gamma_b^{max}, \quad \gamma_b' = \Gamma - \gamma_a^{max},$$

$$q^- = \frac{\gamma_b'}{\gamma_a^{max}}, \quad q^+ = \frac{\gamma_b^{max}}{\gamma_a'},$$

then, for $p \ge \frac{1}{1+q^-}$, **B1** is false and **B2** is true, for $p \le \frac{1}{1+q^+}$, **B1** is true and **B2** is false and finally, for $\frac{1}{1+q^+} \le p \le \frac{1}{1+q^-}$, both **B1** and **B2** are true.

### A. *Optimal choice for flux of Example 2*

To compute the costs we need to determine the equilibrium densities $\hat{\rho}$. In general, $\hat{\rho}_\varphi = f^{-1}(\hat{\gamma}_\varphi)$ ($\hat{\rho}_\psi = f^{-1}(\hat{\gamma}_\psi)$ resp.) and $\hat{\gamma}_\varphi$ is either $\tilde{\gamma}_\varphi$ or $\gamma_\varphi^{max}$ or $\Gamma - \gamma_\varphi^{max}$ ($\hat{\gamma}_\psi$ is either $\tilde{\gamma}_\psi$ or $\gamma_\psi^{max}$ or $\Gamma - \gamma_\psi^{max}$ resp.).

Let us now focus on the flux of Example 2. For simplicity we set here $v_{max} = \rho_{max} = 1$ hence $v(\rho) = 1 - \rho$ and $f(\rho) = \rho(1-\rho)$. We want to solve $\hat{\rho}(1-\hat{\rho}) = \hat{\gamma}$. Hence, by solving $\hat{\rho}^2 - \hat{\rho} + \hat{\gamma} = 0$, we get $\hat{\rho} = \frac{1}{2}(1 \pm \sqrt{\Delta(\hat{\gamma})})$ where $\Delta = \Delta(\hat{\gamma}) = 1 - 4\hat{\gamma}$ and $v(\hat{\rho}_\varphi) = (1-\hat{\rho}_\varphi) = \frac{1}{2}(1 - s_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)})$ ($v(\hat{\rho}_\psi) = (1-\hat{\rho}_\psi) = \frac{1}{2}(1 - s_\psi\sqrt{\Delta(\hat{\gamma}_\psi)})$ resp.). Define for incoming lines $s_\varphi = -1$ if $\rho_\varphi \le \sigma$ and $\Gamma = \Gamma_{in}$, $s_\varphi = +1$ otherwise; and for outgoing ones $s_\psi = +1$ if $\rho_\varphi \ge \sigma$ and $\Gamma = \Gamma_{out}$, $s_\psi = -1$ otherwise. Then, recalling (6) and (7), we get

$$V_{\varphi\psi} = \tfrac{1}{2}(2 - s_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} - s_\psi\sqrt{\Delta(\hat{\gamma}_\psi)}), \quad J_1 = 4 - \left(s_a\sqrt{\Delta(\hat{\gamma}_a)} + s_b\sqrt{\Delta(\hat{\gamma}_b)} + s_c\sqrt{\Delta(\hat{\gamma}_c)} + s_d\sqrt{\Delta(\hat{\gamma}_d)}\right)$$

and

$$J_2 = (\rho_a + \rho_b) - \tfrac{1}{2}\left(s_a\rho_a\sqrt{\Delta(\hat{\gamma}_a)} + s_b\rho_b\sqrt{\Delta(\hat{\gamma}_b)} + s_c\rho_c\sqrt{\Delta(\hat{\gamma}_c)} + s_d\rho_d\sqrt{\Delta(\hat{\gamma}_d)}\right).$$

Finally we want to maximize the cost $J_1 = 4 - \sum s_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)}$ and $J_2 = (\rho_a + \rho_b) - \frac{1}{2}\sum s_\varphi\rho_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)}$ with respect to $\alpha$ and $p$.
Let us start with $\alpha$. Assume first that $\Gamma = \Gamma_{in} = \Gamma_{out}$. Then both **A1** and **A2** are satisfied if and only if $\beta^- = \beta^+ = \frac{\gamma_d^{max}}{\gamma_c^{max}}$, hence $\alpha = \frac{\gamma_c^{max}}{\gamma_c^{max} + \gamma_d^{max}} = \frac{\gamma_c^{max}}{\Gamma}$. In this case $\hat{\gamma} = (\gamma_a^{max}, \gamma_b^{max}, \gamma_c^{max}, \gamma_d^{max})$, and $J_1$ and $J_2$ do not depend on $\alpha$. The same happens for $\Gamma < \Gamma_{in}$, thus we focus on the case $\Gamma = \Gamma_{in} < \Gamma_{out}$. We have:

$$\hat{\gamma} = (\gamma_a^{max}, \gamma_b^{max}, \alpha(\gamma_a^{max} + \gamma_b^{max}), (1-\alpha)(\gamma_a^{max} + \gamma_b^{max})).$$

Recalling that $\Gamma_{in} = \gamma_a^{max} + \gamma_b^{max}$:

$$J_1 = 4 - \sum s_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} =$$
$$4 - (s_a\sqrt{1 - 4\gamma_a^{max}} + s_b\sqrt{1 - 4\gamma_b^{max}}) \tag{10}$$
$$-(s_c\sqrt{1 - 4\alpha\Gamma_{in}} + s_d\sqrt{1 - 4(1-\alpha)\Gamma_{in}}) \tag{11}$$

and

$$J_2 = (\rho_a + \rho_b) - \sum s_\varphi\rho_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} =$$
$$(\rho_a + \rho_b) - (s_a\rho_a\sqrt{1 - 4\gamma_a^{max}} + s_b\rho_b\sqrt{1 - 4\gamma_b^{max}}) \tag{12}$$
$$-(s_c\rho_c\sqrt{1 - 4\alpha\Gamma_{in}} + s_d\rho_d\sqrt{1 - 4(1-\alpha)\Gamma_{in}}). \tag{13}$$

Now the part of the cost in (10) and in (12) does not depend on $\alpha$ and maximizing $J_1$ and $J_2$ is equivalent to maximizing expressions (11) and (13). Since we are in the case $\Gamma = \Gamma_{in} < \Gamma_{out}$, we get $s_c = s_d = -1$. Hence we have to maximize the expressions

$$\begin{aligned} \hat{J}_1 &= \sqrt{1-4\alpha\Gamma_{in}} + \sqrt{1-4(1-\alpha)\Gamma_{in}} && (14)\\ \hat{J}_2 &= \rho_c\sqrt{1-4\alpha\Gamma_{in}} + \rho_d\sqrt{1-4(1-\alpha)\Gamma_{in}}. && (15)\end{aligned}$$

Now the case $\rho_a = \rho_b = \frac{1}{2}$ cannot happen since we would have $\gamma_a^{max} = \gamma_b^{max} = \frac{1}{4}$, and $\Gamma = \frac{1}{2}$. But the maximal value of $\Gamma_{out}$ is $\frac{1}{2}$ which fact contradicts the assumption that $\Gamma_{in} < \Gamma_{out}$. Assume then that not both $\rho_a$ and $\rho_b$ are equal to $\frac{1}{2}$. For $\alpha = 0, 1$ we get $\hat{J}_1(0) = \hat{J}_1(1) = \sqrt{1-4\Gamma_{in}}$ and for $\alpha = \frac{1}{2}$ we get $\hat{J}_1(\frac{1}{2}) = 2\sqrt{1-2\Gamma_{in}}$. Hence, since $\Gamma \leq \frac{1}{2}$, we obtain $\hat{J}_1(0) < \hat{J}_1\left(\frac{1}{2}\right)$.
Moreover

$$\frac{\partial}{\partial\alpha}\hat{J}_1(\alpha) = 2\Gamma_{in}\frac{-\sqrt{1-4(1-\alpha)\Gamma_{in}} + \sqrt{1-4\alpha\Gamma_{in}}}{\sqrt{1-4\alpha\Gamma_{in}}\sqrt{1-4(1-\alpha)\Gamma_{in}}} > 0$$

for $\alpha < \frac{1}{2}$. Then the cost function $J_1$ is maximized for the smallest or the biggest value of $\alpha$ which guarantees conditions **A1** and **A2**.
For $\alpha = 0$ we get $\hat{J}_2(0) = \rho_d\sqrt{1-4\Gamma_{in}}$, for $\alpha = 1$ we get $\hat{J}_2(1) = \rho_c\sqrt{1-4\Gamma_{in}}$,

$$\frac{\partial}{\partial\alpha}J_2 = 2\Gamma_{in}\frac{-\rho_c\sqrt{1-4(1-\alpha)\Gamma_{in}} + \rho_d\sqrt{1-4\alpha\Gamma_{in}}}{\sqrt{1-4\alpha\Gamma_{in}}\sqrt{1-4(1-\alpha)\Gamma_{in}}} > 0$$

for $\alpha < \bar{\alpha} = \frac{\rho_d^2 - \rho_c^2(1-4\Gamma_{in})}{4\Gamma_{in}(\rho_d^2+\rho_c^2)}$, and $\hat{J}_2(\bar{\alpha}) = \sqrt{\rho_c^2+\rho_d^2}\sqrt{2(1-2\Gamma_{in})}$. Then $J_2$ is maximized for the smallest or the biggest value of $\alpha$ which guarantees conditions **A1** and **A2**.
Let us now consider the cases where **A2** is satisfied but **A1** is not and viceversa. In this case we either have $\hat{\gamma} = (\gamma_a^{max}, \gamma_b^{max}, \gamma_c^{max}, \Gamma - \gamma_c^{max})$ with

$$\begin{aligned} &J_1 = 4 - \sum s_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} = 4 - (s_a\sqrt{1-4\gamma_a^{max}} + s_b\cdot\\ &\sqrt{1-4\gamma_b^{max}}) - (s_c\sqrt{1-4\gamma_c^{max}} + s_d\sqrt{1-4(\Gamma-\gamma_c^{max})})\end{aligned}$$

and

$$\begin{aligned} &J_2 = (\rho_a+\rho_b) - \sum s_\varphi\rho_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} =\\ &(\rho_a+\rho_b) - (s_a\rho_a\sqrt{1-4\gamma_a^{max}} + s_b\rho_b\sqrt{1-4\gamma_b^{max}})\\ &-(s_c\rho_c\sqrt{1-4\gamma_c^{max}} + s_d\rho_d\sqrt{1-4(\Gamma-\gamma_c^{max})})\end{aligned}$$

or $\hat{\gamma} = (\gamma_a^{max}, \gamma_b^{max}, \Gamma - \gamma_d^{max}, \gamma_d^{max})$ with

$$\begin{aligned} &J_1 = 4 - \sum s_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} = 4 - (s_a\sqrt{1-4\gamma_a^{max}} + s_b\cdot\\ &\sqrt{1-4\gamma_b^{max}}) - (s_d\sqrt{1-4\gamma_d^{max}} + s_c\sqrt{1-4(\Gamma-\gamma_d^{max})}\end{aligned}$$

and

$$\begin{aligned} &J_2 = (\rho_a+\rho_b) - \sum s_\varphi\rho_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} =\\ &(\rho_a+\rho_b) - (s_a\rho_a\sqrt{1-4\gamma_a^{max}} + s_b\rho_b\sqrt{1-4\gamma_b^{max}})\\ &-(s_d\rho_d\sqrt{1-4\gamma_d^{max}} + s_c\rho_c\sqrt{1-4(\Gamma-\gamma_d^{max})}).\end{aligned}$$

Clearly we have that $J_1$ and $J_2$ do not depend on $\alpha$.

### B. Optimal choice of $\alpha$ for Example 2

We can collect the information above as follows:
- $\beta^- \leq 1 \leq \beta^+$ then $J_1$ is constant for $0 \leq \alpha \leq \frac{1}{1+\beta^+}$ then it increases for $\frac{1}{1+\beta^+} \leq \alpha \leq \frac{1}{2}$ then it decreases for $\frac{1}{2} \leq \alpha \leq \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \leq \alpha \leq 1$. The optimal value for $J_1$ is for $\alpha = \frac{1}{2}$.
- $\beta^- \leq \beta^+ \leq 1$ then $J_1$ is constant for $0 \leq \alpha \leq \frac{1}{1+\beta^+}$ then it decreases for $\frac{1}{1+\beta^+} \leq \alpha \leq \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \leq \alpha \leq 1$. The optimal values of $J_1$ are for $\alpha \in [0, \frac{1}{1+\beta^+}]$.
- $1 \leq \beta^- \leq \beta^+$ then $J_1$ is constant for $0 \leq \alpha \leq \frac{1}{1+\beta^+}$ then it increases for $\frac{1}{1+\beta^+} \leq \alpha \leq \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \leq \alpha \leq 1$. The optimal values of $J_1$ are for $\alpha \in [\frac{1}{1+\beta^-}, 1]$.

Analogously for $J_2$ we have the following cases:
- $\beta^- \leq \frac{1}{\bar{\alpha}} - 1 \leq \beta^+$ then $J_2$ is constant for $0 \leq \alpha \leq \frac{1}{1+\beta^+}$ then it increases for $\frac{1}{1+\beta^+} \leq \alpha \leq \bar{\alpha}$ then it decreases for $\bar{\alpha} \leq \alpha \leq \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \leq \alpha \leq 1$. The optimal value for $J_2$ is for $\alpha = \bar{\alpha}$.
- $\beta^- \leq \beta^+ \leq \frac{1}{\bar{\alpha}} - 1$ then $J_2$ is constant for $0 \leq \alpha \leq \frac{1}{1+\beta^+}$ then it decreases for $\frac{1}{1+\beta^+} \leq \alpha \leq \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \leq \alpha \leq 1$. The optimal values of $J_2$ are for $\alpha \in [0, \frac{1}{1+\beta^+}]$.
- $\frac{1}{\bar{\alpha}} - 1 \leq \beta^- \leq \beta^+$ then $J_2$ is constant for $0 \leq \alpha \leq \frac{1}{1+\beta^+}$ then it increases for $\frac{1}{1+\beta^+} \leq \alpha \leq \frac{1}{1+\beta^-}$ and then it is again constant for $\frac{1}{1+\beta^-} \leq \alpha \leq 1$. The optimal values of $J_2$ are for $\alpha \in [\frac{1}{1+\beta^-}, 1]$.

### C. Optimal choice of $p$ for Example 2

The parameter $p$ affects the costs only if $\Gamma = \Gamma_{out} < \Gamma_{in}$. In this case we have $\hat{\gamma} = (p\Gamma, (1-p)\Gamma, \gamma_c^{max}, \gamma_d^{max})$ and:

$$\begin{aligned} &J_1 = 4 - \sum s_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} =\\ &-(s_a\sqrt{1-4p\Gamma} + s_b\sqrt{1-4(1-p)\Gamma}) && (16)\\ &+4 - (s_c\sqrt{1-4\gamma_c^{max}} + s_d\sqrt{1-4\gamma_d^{max}}) && (17)\end{aligned}$$

and

$$\begin{aligned} &J_2 = (\rho_a+\rho_b) - \sum s_\varphi\rho_\varphi\sqrt{\Delta(\hat{\gamma}_\varphi)} =\\ &-(s_a\rho_a\sqrt{1-4p\Gamma} + s_b\rho_b\sqrt{1-4(1-p)\Gamma}) && (18)\\ &+(\rho_a+\rho_b) - (s_c\rho_c\sqrt{1-4\gamma_c^{max}} + s_d\rho_d\sqrt{1-4\gamma_d^{max}}). && (19)\end{aligned}$$

Now the part of the cost in (17) and in (19) does not depend on $p$ and maximizing $J_1$ and $J_2$ is equivalent to maximizing expressions (16) and (18).
Since we are in the case $\Gamma = \Gamma_{out} < \Gamma_{in}$, we get $s_c = s_d = +1$ and the maximization of $J_1$, $J_2$ is equivalent to maximize

$$\begin{aligned} \hat{J}_1 &= -(\sqrt{1-4p\Gamma} + \sqrt{1-4(1-p)\Gamma}) && (20)\\ \hat{J}_2 &= -(\rho_a\sqrt{1-4p\Gamma} + \rho_b\sqrt{1-4(1-p)\Gamma}). && (21)\end{aligned}$$

Notice that in this case we have the same expressions of $\hat{J}_1$ and $\hat{J}_2$ that we obtained in equations (14) and (15) with opposite sign and $p$ in the place of $\alpha$. The analysis thus follows easily. In particular when conditions **B1** is false and **B2** is true or viceversa then the cost functions $J_1$ and $J_2$ are constant with respect to $p$.

We can collect the information on $p$ in the following way:
• $q^- \leq 1 \leq q^+$ then $J_1$ is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it decreases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{2}$ then it increases for $\frac{1}{2} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant forWe distinguish three cases:
$\frac{1}{2} - \frac{1}{1+q^+} > \frac{1}{1+q^-} - \frac{1}{2}$,
$\frac{1}{2} - \frac{1}{1+q^+} = \frac{1}{1+q^-} - \frac{1}{2}$ and
$\frac{1}{2} - \frac{1}{1+q^+} < \frac{1}{1+q^-} - \frac{1}{2}$.
Simplifying we obtain the three cases: $q^-q^+ > 1$, $q^-q^+ = 1$ and $q^-q^+ < 1$. In the first case we have that the optimal values of $J_1$ are for $p \in [0, \frac{1}{1+\beta^+}]$, in the second case the optimal values of $J_1$ are for $p \in [0, \frac{1}{1+q^+}[ \cup ]\frac{1}{1+q^-}, 1]$, and in the third case the optimal values of $J_1$ are for $p \in [\frac{1}{1+q^-}, 1]$,
• $q^- \leq q^+ \leq 1$ then $J_1$ is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it increases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$. The optimal values of $J_1$ are for $p \in [\frac{1}{1+q^-}, 1]$.
• $1 \leq q^- \leq q^+$ then $J_1$ is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it decreases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$.
The optimal values of $J_1$ are for $p \in [0, \frac{1}{1+q^+}]$.
Analogously for $J_2$ we have the following cases:
• $q^- \leq \frac{1}{\bar{p}} - 1 \leq q^+$ then $J_2$ is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it decreases for $\frac{1}{1+q^+} \leq p \leq \bar{p}$ then it increases for $\bar{p} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$. We distinguish three cases:
$\bar{p} - \frac{1}{1+q^+} > \frac{1}{1+q^-} - \bar{p}$,
$\bar{p} - \frac{1}{1+q^+} = \frac{1}{1+q^-} - \bar{p}$ and
$\bar{p} - \frac{1}{1+q^+} < \frac{1}{1+q^-} - \bar{p}$.
Simplifying we obtain $\frac{1}{1+q^-} + \frac{1}{1+q^+} < 2\bar{p}$, $\frac{1}{1+q^-} + \frac{1}{1+q^+} = 2\bar{p}$ and $\frac{1}{1+q^-} + \frac{1}{1+q^+} > 2\bar{p}$. In the first case we have that the optimal values of $J_2$ are for $p \in [0, \frac{1}{1+q^+}]$, in the second case the optimal values of $J_1$ are for $p \in [0, \frac{1}{1+q^+}[ \cup ]\frac{1}{1+q^-}, 1]$, and in the third case the optimal values of $J_2$ are for $p \in [\frac{1}{1+q^-}, 1]$,
• $q^- \leq q^+ \leq \frac{1}{\bar{p}} - 1$ then $J_2$ is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it increases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$. The optimal values of $J_2$ are for $p \in [\frac{1}{1+q^-}, 1]$.
• $\frac{1}{\bar{p}} - 1 \leq q^- \leq q^+$ then $J_2$ is constant for $0 \leq p \leq \frac{1}{1+q^+}$ then it decreases for $\frac{1}{1+q^+} \leq p \leq \frac{1}{1+q^-}$ and then it is again constant for $\frac{1}{1+q^-} \leq p \leq 1$. The optimal values of $J_2$ are for $p \in [0, \frac{1}{1+q^+}]$.

### D. Optimal choices for Example 1

Notice that, in this case, the velocity $v(\rho)$ is constant for $\rho < \sigma$. Hence if $\Gamma = \Gamma_{in} < \Gamma_{out}$, i.e. the parameter $\alpha$ is involved, then all solutions produce the same velocity thus the same costs $J_1$ and $J_2$.
It remain to discuss the other case where $\Gamma = \Gamma_{out} < \Gamma_{in}$ and only the parameter $p$ matters. We get

$$\frac{\partial J_1}{\partial p} = \frac{(2p-1)\Gamma^2(2-\Gamma)}{(1-p\Gamma)(1-(1-p)\Gamma)}$$

thus the sign is determined by the term $(2p-1)$ which is negative for $p \leq \frac{1}{2}$ and positive for $p \geq \frac{1}{2}$. We obtain the same conclusions as in the previous subsection.

## VI. Conclusions

We considered a macroscopic model for packets flow on a telecommunication network obtained by looking at intermediate time scale.
After the definition of solutions to Riemann problems at nodes by a routing algorithm, we introduced some functions measuring the average velocity of packets on the network. For a simple network, we optimized the choice of parameters involved in the solutions of Riemann problems to maximize such performance functions.

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